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The effect of a weak heterogeneity of a porous medium on natural convection

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The results of an investigation on the effect of a weak heterogeneity of a porous medium on natural convection are presented. A medium heterogeneity is represented by spatial variations of the permeability and of the effective thermal conductivity. As a general rule the existence of horizontal thermal gradients in heterogeneous porous media provides a sufficient condition for the occurrence of natural convection. The implications of this condition are investigated for horizontal layers or rectangular domains subject to isothermal top and bottom boundary conditions. Results lead to a restriction on the classes of thermal conductivity functions which allow a motionless solution. Analytical solutions for rectangular weak heterogeneous porous domains heated from below, consistent with a basic motionless solution, are obtained by applying the weak nonlinear theory. The amplitude of the convection is obtained from an ordinary non-homogeneous differential equation, with a forcing term representative of the medium heterogeneity with respect to the effective thermal conductivity. A smooth transition through the critical Rayleigh number is obtained, thus removing a bifurcation which usually appears in homogeneous domains with perfect boundaries, at the critical value of the Rayleigh number. Within a certain range of slightly supercritical Rayleigh numbers, a symmetric thermal conductivity function is shown to reinforce a symmetrical flow while antisymmetric functions favour an antisymmetric flow. Except for the higher-order solutions, the weak heterogeneity with respect to permeability plays a relatively passive role and does not affect the solutions at the leading order. In contrast, the weak heterogeneity with respect to the effective thermal conductivity does have a significant effect on the resulting flow pattern.

1. Introduction

Natural convection in porous media is of practical interest in geophysics and engineering. Heat transfer in geothermal systems and insulation technology serve as typical examples. In geophysical and engineering applications the porous domain is frequently non-homogeneous in the sense that its permeability and effective thermal conductivity exhibit spatial dependence. For a pure fluid (non-porous domain) thermal conductivity is at most temperature dependent (but not explicitly spatially dependent). In contrast, thermal conductivity of a porous domain can be explicitly spatially dependent because of variations in the porosity and the solid phase composition. Practically every porous domain exhibits at least weak heterogeneity with respect to permeability and effective thermal conductivity. Therefore the influence of heterogeneity on natural convection is of practical interest in the aforementioned geophysical and engineering applications.

This significance continued to be recognized by scientists and engineers. For example Gheorghitza (1961) considered a porous domain consisting of two homogeneous horizontal layers having different permeabilities. McKibbin & O'Sullivan (1980) extended this case to a multilayered stratification in a closed domain and concluded that large permeability differences between the layers are required to force the system into an onset-mode quite different from a homogeneous system. Although their equations included the possibility of a heterogeneous thermal conductivity, the isolated effect of this factor was not presented. Rubin (1981) investigated the heterogeneity effect in a porous domain with vertical variations in permeability and effective thermal conductivity. He obtained a condition for the existence of a motionless solution as a restriction on the basic temperature distribution. The investigation here shows that Rubin's (1981) restriction is valid only for the particular case of vertical stratification. For the more general case further restrictions are required. Gjerde & Tyvand (1984) and Nield (1987) considered vertical or horizontal and continuous or discrete stratification but excluded the possible variation of the effective thermal conductivity. Such variations were investigated by McKibbin (1986) for a horizontal stratification consisting of vertical columns.

All these studies considering heterogeneity assume a vertical or horizontal stratification, i.e. the division of the horizontal domain into horizontal sub-layers or vertical columns having different values of permeability or effective thermal conductivity. The majority of these cases are limited to investigating the effect of spatially dependent permeability but for a constant value of effective thermal conductivity. Here we consider a general form of stratification of the porous-medium permeability and thermal conductivity and investigate its effect on natural convection. Although our primary interest is the steady-state solutions to the problem, we retain the time dependence in order to discuss the stability of the solutions, and the interaction between the initial conditions and the inherent imperfection of the heterogeneous porous medium. Similar effects of small imperfections on the convection at the leading order were obtained for boundary imperfections by O'Sullivan & McKibbin (1986) and by Vadasz & Braester (1992). In contrast, the present investigation considers an inherent rather than a boundary imperfection.

2. Mathematical formulation

Consider a fluid-saturated heterogeneous porous domain confined by rigid boundaries. The flow is assumed to be in the range where Darcy's law applies. At each point of the flow domain the temperatures of the solid and fluid phases are assumed to be equal. Under Boussinesq's approximation (see for instance Dagan 1972, pp. 55–64) and assuming a linear relation between the density and temperature, the governing mass, momentum and energy balance equations, expressed in a dimensionless form, are given by

$$\nabla \cdot \boldsymbol{q} = 0, \tag{1}$$

$$\boldsymbol{q} = -k(\nabla p + Ra T \hat{\boldsymbol{e}}_g), \tag{2}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{q} \cdot \nabla T = \nabla \cdot (\lambda \nabla T), \tag{3}$$

where Ra is the porous-media Rayleigh number defined as

$$Ra = \beta_* \Delta T_c g_* k_0 l_c M_f / \alpha_{e0} \nu_*$$

in which an asterisk denotes dimensional quantities, M_f is the ratio between the heat capacity of the fluid and the effective heat capacity of the porous medium domain, β_* is the thermal expansion coefficient, g_* is the gravity acceleration, k_0 is a permeability reference value, ΔT_c denotes a characteristic temperature difference, l_c is a characteristic lengthscale, α_{e0} is the reference value of the effective thermal diffusivity of the fluid-saturated porous domain and ν_* is the fluid kinematic viscosity. In (1), (2), and (3) $\boldsymbol{q}$ is Darcy's flux, T is the temperature and p is the pressure related to an adiabatic hydrostatic reference, $\hat{\boldsymbol{e}}_g$ is a unit vector in the direction of gravity, and k and λ are the permeability and the effective thermal conductivity functions of the porous domain, respectively. The heterogeneity of the porous medium is defined by k and λ which are allowed to vary within the domain. Yet k and λ are assumed independent of temperature. The variables are scaled by using l_c, $(l_c^2 M_f)/\alpha_{e0}$, $\alpha_{e0}/(l_c M_f)$, $(\mu_* \alpha_{e0})/(k_0 M_f)$ and ΔT_c as the characteristic values of length, time, Darcy's flux, pressure and temperature variations, respectively. A reference value of permeability, k_0, is used to scale the permeability function $k_*(\boldsymbol{X})$. Similarly a reference for the effective thermal conductivity, λ_{e0}, is used to scale the conductivity function, $\lambda_{e*}(\boldsymbol{X})$, where $\boldsymbol{X}$ denotes the position vector $(= x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z$, where $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$ and $\hat{\boldsymbol{e}}_z$ are unit vectors in the directions x, y, and z respectively). The effective thermal conductivity function is defined as $\lambda_e = \varphi\lambda_f + (1-\varphi)\lambda_s$, where φ is the porosity and λ_f and λ_s are the values of thermal conductivity in the fluid and solid phases, respectively.

The following analysis is confined to closed domains D, bounded by impermeable boundaries B. This type of boundary implies the boundary condition $\boldsymbol{q}\cdot\hat{\boldsymbol{e}}_n|_B = 0$, where $\hat{\boldsymbol{e}}_n$ is a unit vector normal to B.

2.1. *A condition for the existence of a motionless solution in heterogeneous porous media*

A Cartesian coordinate system is used such that the vertical axis z, positive upwards, is collinear with gravity; then in (2) $\hat{\boldsymbol{e}}_g = -\hat{\boldsymbol{e}}_z$. Dividing (2) by k and applying the curl operator results in

$$\nabla\times(T\hat{\boldsymbol{e}}_z) \equiv \nabla T\times\hat{\boldsymbol{e}}_z = 0 \quad \text{for} \quad \boldsymbol{q} = 0. \tag{4}$$

This condition implies that

$$\boldsymbol{q} = 0 \Rightarrow \nabla_H T = 0, \tag{5}$$

where $\nabla_H T = (\partial T/\partial x)\hat{\boldsymbol{e}}_x + (\partial T/\partial y)\hat{\boldsymbol{e}}_y$ is the horizontal thermal gradient. Thus the necessary condition for the existence of a motionless state in a heterogeneous porous domain is similar to the corresponding condition for a homogeneous porous medium, i.e. $\nabla_H T = 0$ throughout the domain. The implications of this condition for horizontal layers or rectangular domains are treated by evaluating the temperature distribution corresponding to a motionless state, i.e. the conduction solution in a heterogeneous medium. For steady-state conditions, i.e. $\partial T/\partial t = 0$ and for a motionless solution, (3) reduces to

$$\nabla\cdot[\lambda\nabla T] = 0. \tag{6}$$

As $\nabla T = \nabla_H T + \partial T/\partial z\hat{\boldsymbol{e}}_z$, the motionless state condition (5) yields $\nabla T = \partial T/\partial z\hat{\boldsymbol{e}}_z$ for $\boldsymbol{q} = 0$. Substitution of this result in (6) and integrating leads to the following solution for heterogeneous media:

$$T = C_0\int\frac{\mathrm{d}z}{\lambda} + C_1 \quad \text{for} \quad \boldsymbol{q} = 0, \tag{7}$$

where the coefficients C_0 and C_1 are to be determined from the boundary conditions and are in general functions of x and y as $\lambda = \lambda(x,y,z)$. For a horizontal layer or a rectangular domain heated from below, the isothermal boundary conditions at the bottom and top are $T]_{z=0} = 1$ and $T]_{z=1} = 0$. The temperature distribution is obtained by evaluating the coefficients C_0, C_1. Equation (7) can then be rewritten as

$$T = \int_z^1 \frac{d\zeta}{\lambda} \Big/ \int_0^1 \frac{d\zeta}{\lambda} = 1 - \int_0^z \frac{d\zeta}{\lambda} \Big/ \int_0^1 \frac{d\zeta}{\lambda} \quad \text{for} \quad q = 0, \tag{8}$$

where ζ is a dummy variable. Since λ is a function of x, y and z it becomes necessary to verify the compatibility of the solution (8) with the motionless condition (5). Substituting (8) into the condition $\nabla_H T = 0$ the requirement for the existence of a motionless solution is obtained as

$$\int_0^z \frac{d\zeta}{\lambda} \int_0^1 \frac{\nabla_H \lambda}{\lambda^2} d\zeta - \int_0^z \frac{\nabla_H \lambda}{\lambda^2} d\zeta \int_0^1 \frac{d\zeta}{\lambda} = 0. \tag{9}$$

The integral condition (9) can be expressed in the following form:

$$\int_0^z d\zeta \left[\frac{1}{\lambda} \int_0^1 \frac{\nabla_H \lambda}{\lambda^2} d\zeta - \frac{\nabla_H \lambda}{\lambda^2} \int_0^1 \frac{d\zeta}{\lambda} \right] = 0. \tag{10}$$

As the integral in (10) must vanish for every value of $z \in [0, 1]$, the integrand itself must necessarily vanish. This leads to

$$\frac{\nabla_H \lambda}{\lambda} = \int_0^1 \frac{\nabla_H \lambda}{\lambda^2} d\zeta \Big/ \int_0^1 \frac{d\zeta}{\lambda}. \tag{11}$$

The equality (11) represents the necessary condition for the motionless condition (9) to hold. Furthermore, it can be presented as two scalar equalities as follows:

$$\frac{\partial(\ln \lambda)}{\partial x} = \int_0^1 \frac{(\partial\lambda/\partial x)}{\lambda^2} d\zeta \Big/ \int_0^1 \frac{d\zeta}{\lambda} = F_1(x,y), \tag{12a}$$

$$\frac{\partial(\ln \lambda)}{\partial y} = \int_0^1 \frac{(\partial\lambda/\partial y)}{\lambda^2} d\zeta \Big/ \int_0^1 \frac{d\zeta}{\lambda} = F_2(x,y). \tag{12b}$$

Therefore

$$\ln \lambda = G(z) + H(x,y), \tag{13}$$

which leads to the necessary condition required for consistency with a motionless solution restricted by the class of functions $\lambda(x,y,z)$. This restriction is expressed by the following form of separation of variables:

$$\lambda(\boldsymbol{X}) = f(z)\, h(x,y), \tag{14}$$

where $f(z)$ and $h(x,y)$ are arbitrary functions of z and (x,y) only. Whenever the effective thermal conductivity function $\lambda(\boldsymbol{X})$ satisfies (14), a motionless state is possible. If this rule is not satisfied natural convection occurs unconditionally. Upon substitution of (14) into the basic conduction solution (8) the temperature distribution takes the form

$$T(z) = 1 - \int_0^z f^{-1}(\zeta)\, d\zeta \Big/ \int_0^1 f^{-1}(\zeta)\, d\zeta \quad \text{for} \quad q = 0, \tag{15}$$

where $f(z)$ is the component of the thermal conductivity function (14) which represents the vertical variations of λ. Despite the dependence of λ on x, y and z it is clear from

(15) that the temperature distribution is a function of z only. This temperature profile is generally nonlinear and may reflect different functional forms according to the particular form of $f(z)$. For example, if $f=$ constant, (15) yields the linear temperature profile $T(z) = 1-z$. In this case the heterogeneity is expressed in the form of a horizontal stratification, i.e. the thermal conductivity is a function of the horizontal coordinates only, $\lambda = h(x,y)$. On the other hand if $h=$ constant in (14), the thermal conductivity depends only on z, i.e. $\lambda = f(z)$, thus leading to a vertical stratification. In both cases the requirement for the existence of a motionless solution, expressed by (14), is satisfied identically. These are the cases investigated in previous studies cited in the introduction. No restriction was found regarding the permeability function as far as the motionless solution is concerned. In order to investigate the effect of a more general stratification, say an arbitrary function $\lambda(x,y,z)$, and not necessarily satisfying (14), we assume a weak heterogeneity with respect to thermal conductivity and permeability.

2.2. *The weak heterogeneous porous medium*

The weak heterogeneity effect described here is confined to a rectangular two-dimensional porous domain subject to isothermal heating from below and cooling from above. Strauss (1974) showed that two-dimensional natural convection is stable up to nine times the value of the critical Rayleigh number. Using a stream function ψ defined by $u = \partial\psi/\partial z$, $w = -\partial\psi/\partial x$, where u and w are the Darcy's flux components in the x- and z-directions respectively, and applying the curl operator on (2) we obtain from (2) and (3)

$$\left[\nabla^2\psi + kRa\frac{\partial T}{\partial x}\right]\hat{e}_y + \nabla k\times\nabla p - RaT\nabla\times(k\hat{e}_z) = 0, \tag{16}$$

$$\lambda\nabla^2 T + \nabla\lambda\cdot\nabla T - \frac{\partial T}{\partial t} - \frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} = 0. \tag{17}$$

Here the operators ∇ and ∇^2 are two-dimensional and represent $\nabla \equiv (\partial/\partial x)\,\hat{e}_x + (\partial/\partial z)\,\hat{e}_z$ and $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial z^2$ respectively. For the rectangular domain the boundary conditions are $\psi = 0 \,\forall X\in B$, $T=1$ at $z=0$, $T=0$ at $z=1$, and $\partial T/\partial x = 0$ at $x=0$ and $x=L$ where L (length/height) is the aspect ratio of the domain. Equations (16) and (17) form a nonlinear coupled system. To obtain an analytical solution we recall that for the corresponding perfect case (homogeneous domain) the linear growth of small perturbations at $Ra > Ra_c$ is proportional to $(Ra - Ra_c)$, where $Ra_c(m,n)$ is the characteristic value of Ra obtained from a linear stability analysis and equals $\pi^2(m^2+n^2L^2)^2/m^2L^2$. This, coupled with the establishment of a balance between the linear growth and decay due to nonlinearity, implies that $(Ra-Ra_c) = O(\epsilon^2)$. Therefore ϵ can be defined by $\epsilon^2 = (Ra-Ra_c)/Ra$ which has a form similar to that used by Palm, Weber & Kvernvold (1972). By using this definition of ϵ the dependent variables ψ, T and p are expanded in the form

$$[\psi, T, p] = [\psi^{(0)}, T^{(0)}, p^{(0)}] + \epsilon[\psi^{(1)}, T^{(1)}, p^{(1)}] + \epsilon^2[\psi^{(2)}, T^{(2)}, p^{(2)}] + \epsilon^3[\psi^{(3)}, T^{(3)}, p^{(3)}] + O(\epsilon^4), \tag{18}$$

where $\psi^{(0)}$, $T^{(0)}$ and $p^{(0)}$ represent the basic motionless solution, i.e. $\psi^{(0)} = 0$, $T^{(0)} = 1-z$ and $p^{(0)} = Ra_c(z - \frac{1}{2}z^2 + \text{const.})$. The Rayleigh number is expanded in a finite power series as follows:

$$Ra = Ra_c + Ra_c^{(2s)}[\epsilon^2 + \epsilon^4 + \dots + \epsilon^{2s}]. \tag{19}$$

For the imperfect case (heterogeneous porous domain) the thermal conductivity and permeability functions λ and k can be expanded in the form

$$\lambda = \lambda_0 + \delta\lambda_1(x,y,z) + O(\delta^2); \quad k = k_0 + \delta k_1(x,y,z) + O(\delta^2), \tag{20}$$

where $0 < \delta \ll 1$ is a small parameter reflecting the weak heterogeneity of the domain. To be consistent with a basic motionless solution λ_0 is constant and without any loss of generality we substitute $\lambda_0 = 1$. In order to obtain a common basis for comparison between the effects of the heterogeneous thermal conductivity and the heterogeneous permeability we set the heterogeneity in k to the same order of λ. Therefore k_0 is a constant and without loss of generality we set $k_0 = 1$. However, the imperfect case depends on two small parameters, the extent of the heterogeneity δ and the thermal forcing (expressed by the deviation of Ra from Ra_c), i.e. ϵ^2. Therefore, an expansion comprising a series in powers of $\delta^i\epsilon^j$ is more appropriate for this case. By using such an expansion we find that for the imperfect linear problem a steady-state solution exists for $Ra < Ra_c$. As $Ra \to Ra_c$ resonance occurs and the amplitude grows like $\delta(Ra - Ra_c)^{-1}$. The singularity is removed by nonlinear effects and a steady state at cubic order exists only if an appropriate balance between $(Ra - Ra_c)^{\frac{1}{2}}$ and $\delta(Ra - Ra_c)^{-1}$ is imposed. This means that as $Ra \to Ra_c$, $\epsilon \sim \delta\epsilon^{-2}$ or $\delta \sim \epsilon^3$. Substituting ϵ^3 for δ the expansion (20) up to order-δ reduces to

$$\lambda = 1 + \epsilon^3\lambda^{(3)}(x,z); \quad k = 1 + \epsilon^3 k^{(3)}(x,z). \tag{21}$$

3. Analytical solution

The basic motionless solutions $\psi^{(0)}$, $T^{(0)}$ and $p^{(0)}$ obey the following hydrostatic relationship resulting from (2):

$$\nabla k \times \nabla p^{(0)} = Ra\, T^{(0)} \nabla \times (k\hat{e}_z). \tag{22}$$

This relationship does not represent any restriction on k. Substitution of (19) and (21) into (22) yields

$$\nabla k^{(3)} \times \nabla p^{(0)} = Ra_c T^{(0)} \nabla \times (k^{(3)}\hat{e}_z). \tag{23}$$

Substitution of the expansions (18) along with the basic solutions $\psi^{(0)} = 0$, $T^{(0)} = 1 - z$ and $p^{(0)} = Ra_c(z - \frac{1}{2}z^2 + \text{const.})$ into the governing equations at order-ϵ leads to

$$\nabla^2\psi^{(1)} + Ra_c \frac{\partial T^{(1)}}{\partial x} = 0, \tag{24}$$

$$\nabla^2 T^{(1)} - \frac{\partial T^{(1)}}{\partial t} - \frac{\partial \psi^{(1)}}{\partial x} = 0. \tag{25}$$

Decoupling the equations and using the homogeneous conditions at this order, i.e.

$$\psi^{(1)} = 0 \;\forall X \in B \quad \text{and} \quad T^{(1)}(x,0) = T^{(1)}(x,1) = \frac{\partial T^{(1)}}{\partial x}(0,z) = \frac{\partial T^{(1)}}{\partial x}(L,z) = 0$$

the solutions at order-ϵ are obtained in the form

$$\psi^{(1)} = A^{(1)}_{mn}(\tau)\sin(m\pi x/L)\sin(n\pi z), \tag{26}$$

$$T^{(1)} = B^{(1)}_{mn}(\tau)\cos(m\pi x/L)\sin(n\pi z). \tag{27}$$

At $Ra = Ra_c$ the linear problem is asymptotically time independent, i.e. the marginally stable mode survives, all other modes decay. Slow time dependence must be restored to eliminate secular terms and thereby extend the range of validity in time to times $t = O(\epsilon^{-2})$. A rescaling of the time variable t is therefore necessary and allows the amplitudes of the solutions at order-ϵ to vary slowly over the large timescale $\tau = \epsilon^2 t$.

Substitution of (26) and (27) into (24) and (25) yields the relationship

$$B^{(1)}_{mn} = -(mL)\,A^{(1)}_{mn}/\pi(m^2+n^2L^2).$$

At this point the amplitudes $A^{(1)}_{mn}$ and $B^{(1)}_{mn}$ remain undetermined; however, their values will be obtained at a later stage using a solvability condition on the equations at order-ϵ^3.

The equations at order-ϵ^2 take the form

$$\nabla^2\psi^{(2)} + Ra_c\frac{\partial T^{(2)}}{\partial x} = 0, \tag{28}$$

$$\nabla^2 T^{(2)} - \frac{\partial\psi^{(2)}}{\partial x} = \frac{\partial\psi^{(1)}}{\partial z}\frac{\partial T^{(1)}}{\partial x} - \frac{\partial\psi^{(1)}}{\partial x}\frac{\partial T^{(1)}}{\partial z}. \tag{29}$$

Upon the substitution of the solutions for $\psi^{(1)}$ and $T^{(1)}$ from (26) and (27) into (29) the right-hand-side forcing functions become

$$\nabla^2 T^{(2)} - \frac{\partial\psi^{(2)}}{\partial x} = -\frac{mn\pi^2}{2L}A^{(1)}_{mn}B^{(1)}_{mn}\sin(2n\pi z). \tag{30}$$

The solution of the system of equations (28) and (30) subject to the homogeneous boundary conditions: $\psi^{(2)} = T^{(2)} = 0$ at $z = 0$ and $z = 1$, and $\psi^{(2)} = \partial T^{(2)}/\partial x = 0$ at $x = 0$ and $x = L$ is obtained by the superposition of a homogeneous and a particular solution. Since the equations and the boundary conditions for the homogeneous part of the system are identical to those solved at order-ϵ, their solution must also be identical. The particular solution introduces an additional term in the expression for $T^{(2)}$ and the complete solutions at order-ϵ^2 are

$$\psi^{(2)} = A^{(2)}_{mn}\sin(m\pi x/L)\sin(n\pi z), \tag{31}$$

$$T^{(2)} = B^{(2)}_{mn}\cos(m\pi x/L)\sin(n\pi z) + B^{(2)}_{0,2n}\sin(2n\pi z), \tag{32}$$

where the relationship between $A^{(2)}_{mn}$ and $B^{(2)}_{mn}$ is given by

$$B^{(2)}_{mn} = -(mL)\,A^{(2)}_{mn}/\pi(m^2+n^2L^2).$$

At this stage the amplitudes $A^{(2)}_{mn}$ and $B^{(2)}_{mn}$ remain undetermined; however, their values can be obtained from the solvability condition of the equations at order-ϵ^4. Similarly the relationship between $B^{(2)}_{0,2n}$ and $A^{(1)}_{mn}$ is given by

$$B^{(2)}_{0,2_n} = -m^2[A^{(1)}_{mn}]^2/8\pi n(m^2+n^2L^2).$$

At order-ϵ^3 one obtains the following set of equations:

$$\left[\nabla^2\psi^{(3)} + Ra_c\frac{\partial T^{(3)}}{\partial x} + Ra^{(2)}_c\frac{\partial T^{(1)}}{\partial x} + k^{(3)}Ra_c\frac{\partial T^{(0)}}{\partial x}\right]\hat{e}_y + \nabla k^{(3)}\times\nabla p^{(0)} - Ra_c\,T^{(0)}\nabla\times(k^{(3)}\hat{e}_z) = 0, \tag{33}$$

$$\nabla^2 T^{(3)} - \frac{\partial\psi^{(3)}}{\partial x} + \left[-\frac{\partial T^{(1)}}{\partial\tau} - \frac{\partial\psi^{(1)}}{\partial z}\frac{\partial T^{(2)}}{\partial x} + \frac{\partial\psi^{(1)}}{\partial x}\frac{\partial T^{(2)}}{\partial z} - \frac{\partial\psi^{(2)}}{\partial z}\frac{\partial T^{(1)}}{\partial x} + \frac{\partial\psi^{(2)}}{\partial x}\frac{\partial T^{(1)}}{\partial z}\right] + \lambda^{(3)}\nabla^2 T^{(0)} + \nabla\lambda^{(3)}\cdot\nabla T^{(0)} = 0. \tag{34}$$

Upon substitution of the basic solution $T^{(0)} = 1-z$ and the relationship (23) into (33) and (34) we obtain

$$\nabla^2\psi^{(3)} + Ra_c\frac{\partial T^{(3)}}{\partial x} = -Ra_c^{(2)}\frac{\partial T^{(1)}}{\partial x}, \tag{35}$$

$$\nabla^2 T^{(3)} - \frac{\partial\psi^{(3)}}{\partial x} = \frac{\partial\lambda^{(3)}}{\partial z} + \left[\frac{\partial T^{(1)}}{\partial\tau} + \frac{\partial\psi^{(1)}}{\partial z}\frac{\partial T^{(2)}}{\partial x} - \frac{\partial\psi^{(1)}}{\partial x}\frac{\partial T^{(2)}}{\partial z} + \frac{\partial\psi^{(2)}}{\partial z}\frac{\partial T^{(1)}}{\partial x} - \frac{\partial\psi^{(2)}}{\partial x}\frac{\partial T^{(1)}}{\partial z}\right]. \tag{36}$$

The corresponding boundary conditions at order-ϵ^3 are $\psi^{(3)} = T^{(3)} = 0$ at $z = 0$ and $z = 1$, and $\psi^{(3)} = \partial T^{(3)}/\partial x = 0$ at $x = 0$ and $x = L$. By substituting (26), (27), (31) and (32) (representing the solutions at orders-ϵ and ϵ^2) into (35) and (36), the right-hand-side forcing functions are evaluated and the equations become

$$\nabla^2\psi^{(3)} + Ra_c\frac{\partial T^{(3)}}{\partial x} = Ra_c^{(2)}\frac{m\pi}{L}B_{mn}^{(1)}\sin\left(\frac{m\pi x}{L}\right)\sin(n\pi z), \tag{37}$$

$$\begin{aligned}\nabla^2 T^{(3)} - \frac{\partial\psi^{(3)}}{\partial x} = {} & \frac{\partial\lambda^{(3)}}{\partial z} - \frac{mn\pi^2}{2L}[A_{mn}^{(1)}B_{mn}^{(2)} + A_{mn}^{(2)}B_{mn}^{(1)}]\sin(2n\pi z) \\ & + \left[\frac{mn\pi^2}{L}A_{mn}^{(1)}B_{0,2n}^{(2)} + \frac{\mathrm{d}B_{mn}^{(1)}}{\mathrm{d}\tau}\right]\cos\left(\frac{m\pi x}{L}\right)\sin(n\pi z) \\ & - \frac{mn\pi^2}{L}A_{mn}^{(1)}B_{0,2n}^{(2)}\cos\left(\frac{m\pi x}{L}\right)\sin(3n\pi z).\end{aligned} \tag{38}$$

As one may observe, (37) and (38) at order-ϵ^3 do not include the heterogeneity with respect to permeability, i.e. $k^{(3)}$. However, as the forcing functions in (38) contain the term $\partial\lambda^{(3)}/\partial z$ the heterogeneity with respect to thermal conductivity is included here. The differential operator of the system of equations (37), (38) is identical to the operator of the equations at order-ϵ. Since (37), (38) at order-ϵ^3 are non-homogeneous versions of the equations at order ϵ, a solvability condition for the equations at order-ϵ^3 must be satisfied. This condition constrains the amplitude of the solution at order-ϵ and enables its determination.

4. The amplitude equation

The solvability condition is derived by multiplying (37) by $\psi^{(1)}$ and (38) by $Ra_c T^{(1)}$, integrating these equations over the domain $x \in [0, L]$, $z \in [0, 1]$ and then adding them. As a result of these operations and making use of Green's second identity, integration by parts, boundary conditions and the results at $O(\epsilon)$, the solvability condition is obtained in the following form:

$$\begin{aligned}\int_0^z \mathrm{d}z\left[-T^{(1)}(0,z)\frac{\partial T^{(3)}}{\partial x}(0,z) + T^{(1)}(L,z)\frac{\partial T^{(3)}}{\partial x}(L,z)\right] = {} & \frac{mn\pi^2}{4}A_{mn}^{(1)}B_{mn}^{(1)}B_{0,2n}^{(2)} \\ & + \frac{Ra_c^{(2)}}{Ra_c}\frac{m\pi}{4}A_{mn}^{(1)}B_{mn}^{(1)} + \frac{B_{mn}^{(1)}L}{4}\frac{\mathrm{d}B_{mn}^{(1)}}{\mathrm{d}\tau} - \int_0^L \mathrm{d}x\int_0^1 \mathrm{d}z\left[\lambda^{(3)}\frac{\partial T^{(1)}}{\partial z}\right].\end{aligned} \tag{39}$$

As the heat flux boundary conditions at the sidewalls are homogeneous, i.e.

$$\partial T^{(3)}/\partial x(0, z) = \partial T^{(3)}/\partial x(L, z) = 0$$

the left-hand-side integral in (39) vanishes. Substituting the $O(\epsilon)$ solution for $T^{(1)}$ from

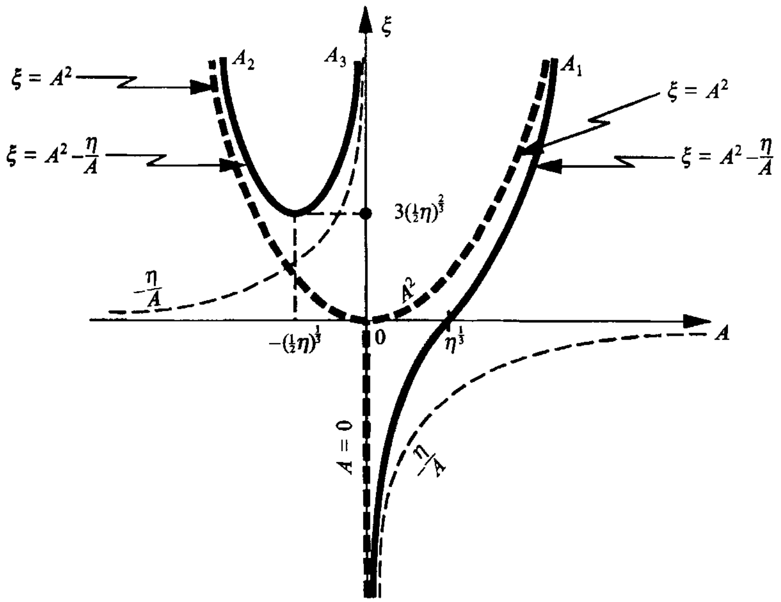

FIGURE 1. Graphical representation of the solutions of the homogeneous and non-homogeneous amplitude equations in the (ξ, A)-plane.

(27) and the relationships between $B^{(1)}_{mn}$ and $A^{(1)}_{mn}$ between $B^{(2)}_{0,2n}$ and $A^{(1)}_{mn}$ in (39), the solvability condition simplifies to

$$\mathrm{d}A/\mathrm{d}t = \chi[\xi A - A^3 + \eta], \tag{40}$$

where the following notation is used:

$$\left.\begin{aligned} \chi = \frac{\pi^2}{8L^2}, \quad A = -\epsilon m A^{(1)}_{mn}, \quad \xi = 8(m^2+n^2L^2)\left[\frac{Ra}{Ra_c} - 1\right], \\ \eta = 32(m^2+n^2L^2)\, n\epsilon^3 \int_0^L \mathrm{d}x\left[\cos\left(\frac{m\pi x}{L}\right)\int_0^1 \mathrm{d}z[\lambda^{(3)}\cos(n\pi z)]\right]. \end{aligned}\right\} \tag{41}$$

In (40) A represents the $O(\epsilon)$ amplitude, ξ is the measure of the deviation of Ra from its critical value ($\xi = 0$ for $Ra = Ra_c$, $\xi < 0$ for $Ra < Ra_c$ and $\xi > 0$ for $Ra > Ra_c$) and η represents the weak $O(\epsilon^3)$ heterogeneity with respect to the effective thermal conductivity. For the particular perfect case (homogeneous porous media) where $\eta = 0$ the steady-state solutions of the amplitude equation (40) are represented graphically in the (ξ, A)-plane by the dashed curves in figure 1. As one may observe, a bifurcation occurs at the critical value of Ra (i.e. at $\xi = 0$). For the imperfect case with a heterogeneous thermal conductivity $\eta \neq 0$ in (40) and for steady-state conditions the following nonlinear algebraic equation results:

$$A^3 - \xi A = \eta. \tag{42}$$

The analytical solution of this equation is

$$A = [\tfrac{1}{2}\eta + (\tfrac{1}{4}\eta^2 - \tfrac{1}{27}\xi^3)^{\frac{1}{2}}]^{\frac{1}{3}} + [\tfrac{1}{2}\eta - (\tfrac{1}{4}\eta^2 - \tfrac{1}{27}\xi^3)^{\frac{1}{2}}]^{\frac{1}{3}} \quad \forall \eta^2 \geqslant \tfrac{4}{27}\xi^3. \tag{43}$$

The validity of the solution (43) is restricted to the condition $\eta^2 \geqslant \frac{4}{27}\xi^3$. This follows from the requirement that the discriminant of (42) is non-negative. In this case a unique real solution of (42) given by (43) exists. For $\eta^2 < 4\xi^3/27$ the discriminant of equation (42) is negative and the following three real solutions are obtained:

$$A_i = 2(\tfrac{1}{3}\xi)^{\frac{1}{2}}\cos(\tfrac{1}{3}\phi + \alpha_i), \quad i = 1,2,3 \quad \forall \eta^2 < \tfrac{4}{27}\xi^3, \tag{44}$$

Case	$\lambda^{(3)}(x,z)$	η
(i)	$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\lambda_{ij}^{cc}\cos\left(\frac{i\pi x}{L}\right)\cos(j\pi z)\right]$	$8(m^2+n^2L^2)\,nL\epsilon^3\lambda_{mn}^{cc}$
(ii)	$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\lambda_{ij}^{cs}\cos\left(\frac{i\pi x}{L}\right)\sin(j\pi z)\right]$	$32(m^2+n^2L^2)\frac{nL}{\pi}\epsilon^3\sum_{\substack{j=1\\(j+n)=\text{odd}}}^{\infty}\lambda_{mj}^{cs}\frac{j}{(j^2-n^2)}$
(iii)	$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\lambda_{ij}^{sc}\sin\left(\frac{i\pi x}{L}\right)\cos(j\pi z)\right]$	$32(m^2+n^2L^2)\frac{nL}{\pi}\epsilon^3\sum_{\substack{i=1\\(i+m)=\text{odd}}}^{\infty}\lambda_{in}^{sc}\frac{i}{(i^2-m^2)}$
(iv)	$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\lambda_{ij}^{ss}\sin\left(\frac{i\pi x}{L}\right)\sin(j\pi z)\right]$	$128(m^2+n^2L^2)\frac{nL}{\pi^2}\epsilon^3\sum_{\substack{i=1\\(i+m)=\text{odd}}}^{\infty}\times\sum_{\substack{j=1\\(j+n)=\text{odd}}}^{\infty}\lambda_{ij}^{ss}\frac{ij}{(i^2-m^2)(j^2-n^2)}$

TABLE 1. The separate contribution of the series groups of terms on η, equation (46)

where ϕ and α_i are defined by

$$\cos\phi=-\tfrac{1}{2}\sqrt{27}\eta/\xi^{\frac{3}{2}},\quad \alpha_i=\begin{cases}0 & \text{for } i=1\\ \frac{2}{3}\pi & \text{for } i=2\\ \frac{4}{3}\pi & \text{for } i=3.\end{cases}\tag{45}$$

One may conclude that for $\xi<3(\frac{1}{2}\eta)^{\frac{2}{3}}$ the amplitude equation has a unique solution. This means that as long as the thermal conductivity heterogeneity is strong enough with respect to ξ, i.e. $|\eta|>2(\frac{1}{3}\xi)^{\frac{3}{2}}$, although still $O(\epsilon^3)$, it controls the intensity and direction of the flow, regardless of the initial flow and temperature conditions. However, if the heterogeneity is weak enough, i.e. $|\eta|<2(\frac{1}{3}\xi)^{\frac{3}{2}}$ there are three possible solutions. The solutions of the amplitude equation (42) are presented in the (ξ, A)-plane in figure 1. The range of the unique solution as well as the range of the three bifurcating solutions corresponding to the three branches A_1, A_2 and A_3 are apparent in the figure. A linear stability analysis of these three solutions was performed using (40). Results show that the solutions A_1 and A_2 are stable while A_3 is unstable. Results also show that in the imperfect case a smooth transition through the critical Rayleigh number replaces the sharp transition that occurred at $Ra_c(\xi=0)$ in the perfect case.

Since η was not given a particular functional form in (41) and the results show that η affects the flow considerably, it is of interest to investigate the effect of general functions $\lambda^{(3)}(x,z)$ on η. For this purpose we expand $\lambda^{(3)}(x,z)$ into a double Fourier series and investigate the isolated effects of groups of terms in the series. The expression for the double Fourier expansion is presented in the Appendix. The constant term λ_{00}^{cc} and the first two series in (A 1) (see Appendix) do not bring any contribution to the amplitude equation (42). This is a direct result of the fact that introduction of these terms in (41) causes the integral to vanish, i.e. one obtains $\eta=0$. Therefore only the terms involved in the double series of (A 1) affects the results.

Actually, a more general conclusion results from the definition of η in (41): any function $\lambda^{(3)}(x,z)$ which is a superposition of functions of x only or z only causes the integral to vanish and does not affect the results, i.e. $\eta=0$. The isolated effects of the groups of terms in the series are presented in table 1. This table shows the results of substituting a series in the form (i)–(iv) into (41) to obtain a corresponding expression for η. The result obtained for case (i) shows that despite the general expansion for $\lambda^{(3)}(x,z)$, which was allowed to include all possible cosine modes, only those modes which reinforce the natural modes of convection, i.e. $i = m, j = n$, affect the amplitude equation through η. The result of case (ii) shows that a weak $O(\epsilon^3)$ heterogeneous thermal conductivity affects the amplitude of convection at order-ϵ at the horizontal natural modes, i.e. $i = m$. In contrast the vertical modes are affected by the value of the sum $(j+n)$, whether even or odd. Thus a symmetric function in the z-direction (with respect to $z = \frac{1}{2}$, i.e. j is odd) gives rise to a symmetric flow in this direction (i.e. an even number of convection cells) and conversely, an antisymmetric function suggests an antisymmetric flow. The results of case (iii) are similar to these obtained for case (ii), the only difference being the exchange of the effects between the x- and z-directions. Similar results can be observed for case (iv) in table 1, i.e. a symmetric function (with respect to $x = \frac{1}{2}L$ or $z = \frac{1}{2}$) suggests a symmetric flow in the same direction and an antisymmetric function an antisymmetric flow. The general series (A 1) is a superposition of the particular cases treated separately. The general expression for η in this case is given by

$$\eta = 8(m^2+n^2L^2)\,nL\epsilon^3\left[\lambda^{cc}_{mn}+\frac{4}{\pi}\sum_{\substack{j=1\\(j+n)=\text{odd}}}^{\infty}\lambda^{cs}_{mj}\frac{j}{(j^2-n^2)}\right.$$

$$\left.+\frac{4}{\pi}\sum_{\substack{i=1\\(i+m)=\text{odd}}}^{\infty}\lambda^{sc}_{in}\frac{i}{(i^2-m^2)}+\frac{16}{\pi^2}\sum_{\substack{i=1\\(i+m)=\text{odd}}}^{\infty}\sum_{\substack{j=1\\(j+n)=\text{odd}}}^{\infty}\lambda^{ss}_{ij}\frac{ij}{(i^2-m^2)(j^2-n^2)}\right]. \quad (46)$$

The conclusions regarding the amplitude of convection and the effect of the separate terms of the series (A 1) have been discussed earlier and apply equally well to (46) above.

5. The mean heat flux

The vertical mean heat flux through the porous domain at steady state, defined in terms of the Nusselt number, is

$$Nu = \frac{1}{L}\int_0^L\left[wT-\lambda\frac{\partial T}{\partial z}\right]\mathrm{d}x. \quad (47)$$

Noting that the integral in (47) is independent of spatial variables the Nusselt number may then be evaluated for convenience at $z = 0$ where $w = 0$. Using (18) and (21) the Nusselt number takes the following form:

$$Nu = \frac{1}{L}\int_0^L-\left[\left(\frac{\partial T^{(0)}}{\partial z}\right)_{z=0}+\epsilon\left(\frac{\partial T^{(1)}}{\partial z}\right)_{z=0}+\epsilon^2\left(\frac{\partial T^{(2)}}{\partial z}\right)_{z=0}\right]\mathrm{d}x+O(\epsilon^3). \quad (48)$$

Substitution of the basic solution for $T^{(0)}$ and for $T^{(1)}$ and $T^{(2)}$ from (27) and (32), respectively, into (48) yields the following expression for the value of Nusselt number to order-ϵ^2:

$$Nu = 1+\frac{1}{4(m^2+n^2L^2)}A^2+O(\epsilon^3), \quad (49)$$

where A is the steady-state solution of the amplitude equation (40), presented in the previous section.

Case	$\lambda(x,z)$	Properties of thermal conductivity function	η_{mn}
(*a*)	$1+\theta_{mn}(x-\frac{1}{2}L)(z-\frac{1}{2})$	Antisymmetric in both *x*- and *z*- directions	$\begin{cases} \dfrac{128(m^2+n^2L^2)\,nL^2}{m^2n^2\pi^4}\theta_{mn} & \forall m = \text{odd}, n = \text{odd} \\ 0 & \forall (m = \text{even}, n = \text{odd}), (m = \text{odd}, n = \text{even}), (m = \text{even}, n = \text{even}) \end{cases}$
(*b*)	$1+\theta_{mn}(x-\frac{1}{2}L)^2(z-\frac{1}{2})^2$	Symmetric in both *x*- *and z*- directions	$\begin{cases} \dfrac{128(m^2+n^2L^2)\,nL^3}{m^2n^2\pi^4}\theta_{mn} & \forall m = \text{even}, n = \text{even} \\ 0 & \forall (m = \text{odd}, n = \text{even}), (m = \text{even}, n = \text{odd}), (m = \text{odd}, n = \text{odd}) \end{cases}$
(*c*)	$1+\theta_{mn}(x-\frac{1}{2}L)^2(z-\frac{1}{2})$	Symmetric in the *x*- direction and antisymmetric in the *z*-direction	$\begin{cases} \dfrac{-128(m^2+n^2L^2)\,nL^3}{m^2n^2\pi^4}\theta_{mn} & \forall m = \text{even}, n = \text{odd} \\ 0 & \forall (m = \text{even}, n = \text{even}), (m = \text{odd}, n = \text{even}), (m = \text{odd}, n = \text{odd}). \end{cases}$
(*d*)	$1+\theta_{mn}(x-\frac{1}{2}L)(z-\frac{1}{2})^2$	Antisymmetric in the *x*- direction and symmetric in the *z*-direction	$\begin{cases} \dfrac{-128(m^2+n^2L^2)\,nL^2}{m^2n^2\pi^4}\theta_{mn} & \forall m = \text{odd}, n = \text{even} \\ 0 & \forall (m = \text{odd}, n = \text{odd}), (m = \text{even}, n = \text{odd}), (m = \text{even}, n = \text{even}). \end{cases}$

TABLE 2. The effect of symmetric and antisymmetric thermal conductivity function on the resulting pattern of convection. ($\theta_{mn} = a\epsilon^3_{mn}$, where a is an arbitrary constant and ϵ_{mn} is the value of ϵ corresponding to Ra_c)

6. Results and discussion

The analytical solutions obtained in the previous sections were used in the evaluation of the amplitude and the Nusselt number corresponding to some particular examples of thermal conductivity functions. These examples are used to demonstrate the effect of symmetric and antisymmetric thermal conductivity functions on the resulting pattern of convection. Four examples were selected corresponding to various combinations of symmetric and antisymmetric functions in the x- and z-directions (with respect to $x = \frac{1}{2}L$, $z = \frac{1}{2}$), as presented in table 2.

In this table $\theta_{mn} = a\epsilon^3_{mn}$ where a is an arbitrary constant and ϵ_{mn} is the value of ϵ corresponding to the characteristic value of *Ra* which depends on m and n. η_{mn} shown in the right-hand column is obtained from $\lambda(x,z)$ by way of (41). It is evident from table 2 that an antisymmetric thermal conductivity function in both the x- and z-directions (example *a*) reinforces antisymmetric solutions, whereas a symmetric function in both the x- and z-directions (example *b*) imposes symmetric solutions. A thermal conductivity function which is symmetric in the x-direction and antisymmetric in the z-direction (example *c*) excites the symmetric modes in the x-direction and the antisymmetric modes in the z-direction. It follows that antisymmetric solutions in the x-direction and symmetric solutions in the z-direction are obtained as a result of a function (example *d*) that is antisymmetric in the x-direction and symmetric in the z-direction.

For the thermal conductivity function corresponding to example (*a*) in table 2 the relationship of η_{mn} was used to evaluate the amplitude A and the Nusselt number for

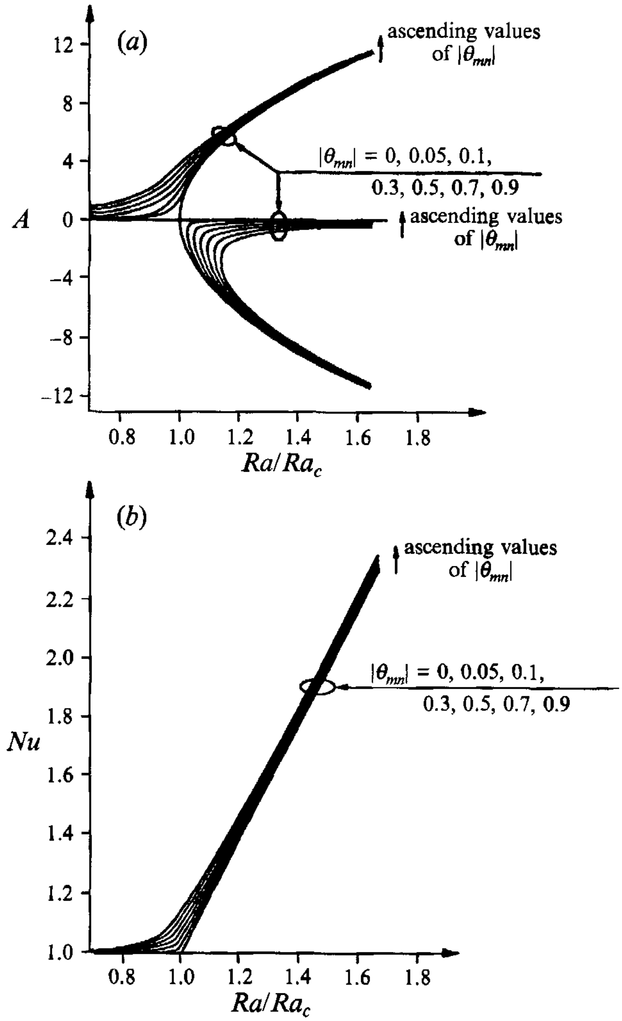

FIGURE 2. The amplitude (*a*) and the Nusselt number (*b*) based on the largest amplitude as function of Ra/Ra_c, for $L = 4$, $m = 3$ and $n = 1$.

different values of Ra and $|\theta_{mn}|$. The amplitude A was evaluated by substituting η_{mn} from table 2 into (43) and (44) for $L = 4$, $m = 3$ and $n = 1$. The results for different values of Ra and $|\theta_{mn}|$ are presented in figure 2(*a*). The corresponding heat flux is presented in terms of Nusselt number (49), in figure 2(*b*). For values of Ra beyond the imperfect bifurcation the largest amplitude was used in (49) to calculate the value of Nu. The smooth transition through the critical value of Ra for non-vanishing values of $|\theta_{mn}|$ is apparent in these figures. The resulting flow and temperature fields corresponding to $L = 4$, $m = 3$ and $n = 1$ are presented in figure 3 for $Ra = 45$. The odd number of convection cells ($m = 3$) is a result of the antisymmetric thermal conductivity function (case *a* in table 2). This solution holds irrespective of the even aspect ratio ($L = 4$). However, for the same value of Rayleigh number, i.e. $Ra = 45$, five convection cells give rise to an additional solution that is consistent with η_{mn} for this case. The resulting flow and temperature fields corresponding to $L = 4$, $m = 5$ and $n = 1$ are presented in figure 4. These two possible solutions are governed by the thermal conductivity function while the flow direction is controlled by the sign of θ_{mn}

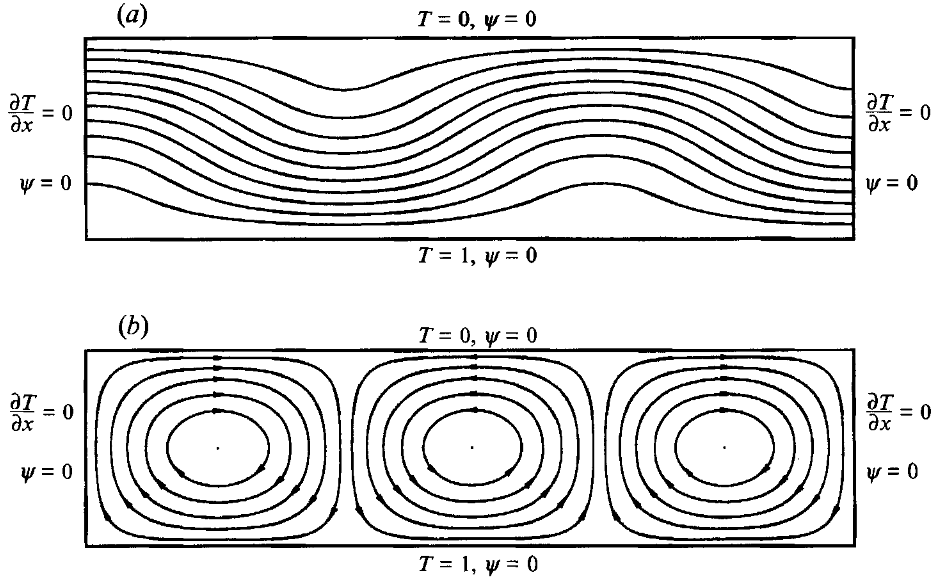

FIGURE 3. Graphical representation of the analytical solutions for the flow and temperature fields corresponding to $\theta_{mn} = 0.86$, $Ra = 45$, $L = 4$, $m = 3$ and $n = 1$. (*a*) 10 isotherms equally divided between $T_{min} = 0$ and $T_{max} = 1$. (*b*) 10 streamlines equally divided between $\psi_{min} = -1.53$ and $\psi_{max} = 1.53$.

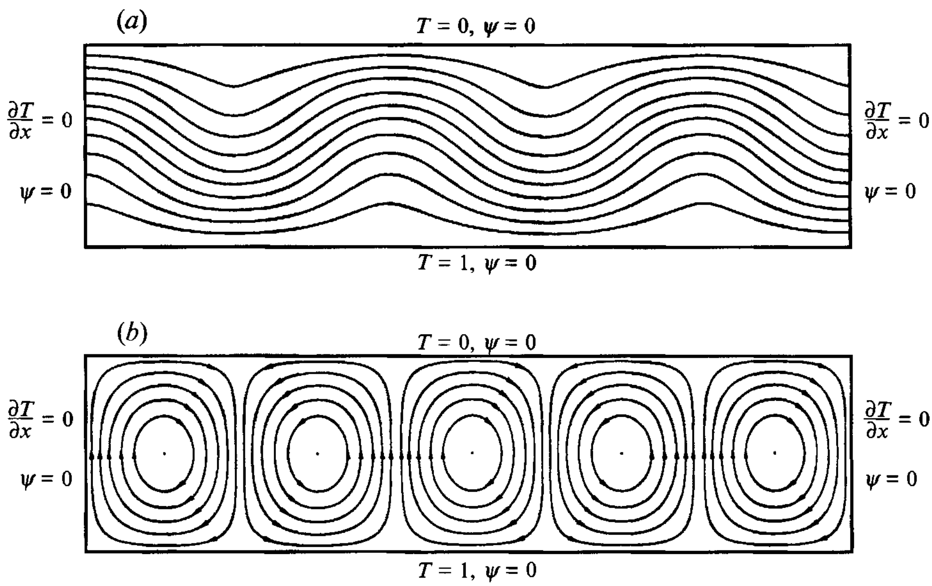

FIGURE 4. As figure 3 but for $\theta_{mn} = 1.72$ and $m = 5$. (*a*) 10 isotherms equally divided between $T_{min} = 0$ and $T_{max} = 1$. (*b*) 10 streamlines equally divided between $\psi_{min} = -1.22$ and $\psi_{max} = 1.22$.

in the expression for $\lambda(x, z)$ in table 2. A four-convection-cell solution is also possible and can be realized by the initial perturbations that include this mode ($m = 4$). In this case the value of the amplitude is obtained from the solution to the homogeneous amplitude equation in which $\eta = 0$.

For the thermal conductivity function corresponding to example (*c*) in table 2, η_{mn} was used to evaluate the amplitude A and the Nusselt number for different values of

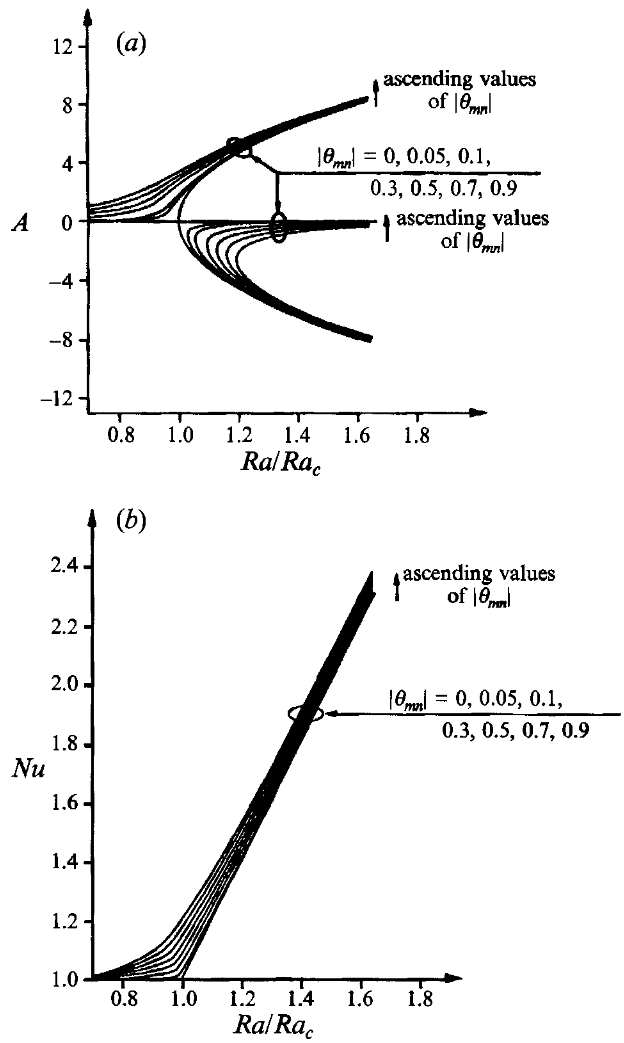

FIGURE 5. The amplitude (*a*) and the Nusselt number (*b*) based on the largest amplitude as function of Ra/Ra_c, for $L = 3$, $m = 2$ and $n = 1$.

Ra and $|\theta_{mn}|$. The amplitude A was evaluated similarly to that in case (*a*) but for $L = 3, m = 2$ and $n = 1$. The results for different values of Ra and $|\theta_{mn}|$ are presented in figure 5(*a*). The corresponding heat flux is presented in terms of the Nusselt number (49), in figure 5(*b*). For values of Ra beyond the imperfect bifurcation the largest amplitude was used in (49) to calculate the value of Nu. For any fixed Rayleigh number and fixed value of $|\theta_{mn}|$ the amplitudes shown in figure 5 are clearly smaller than those shown in figure 2. The resulting flow and temperature fields corresponding to case (*c*) and to $L = 3$, $m = 2$ and $n = 1$ are presented in figure 6 for $Ra = 50$. The even number of convection cells ($m = 2$) is a result of the thermal conductivity function (case *c* in table 2). Here symmetry of the function in the x-direction imposes this pattern despite the odd aspect ratio ($L = 3$). Nevertheless, for the same value of Rayleigh number, i.e. $Ra = 50$, four convection cells represent an additional solution which is consistent with η_{mn} for this case. The resulting flow and temperature field corresponding to $L = 3$, $m = 4$ and $n = 1$ are presented in figure 7. These two possible solutions are governed by the thermal conductivity function while the flow direction is controlled by the sign of

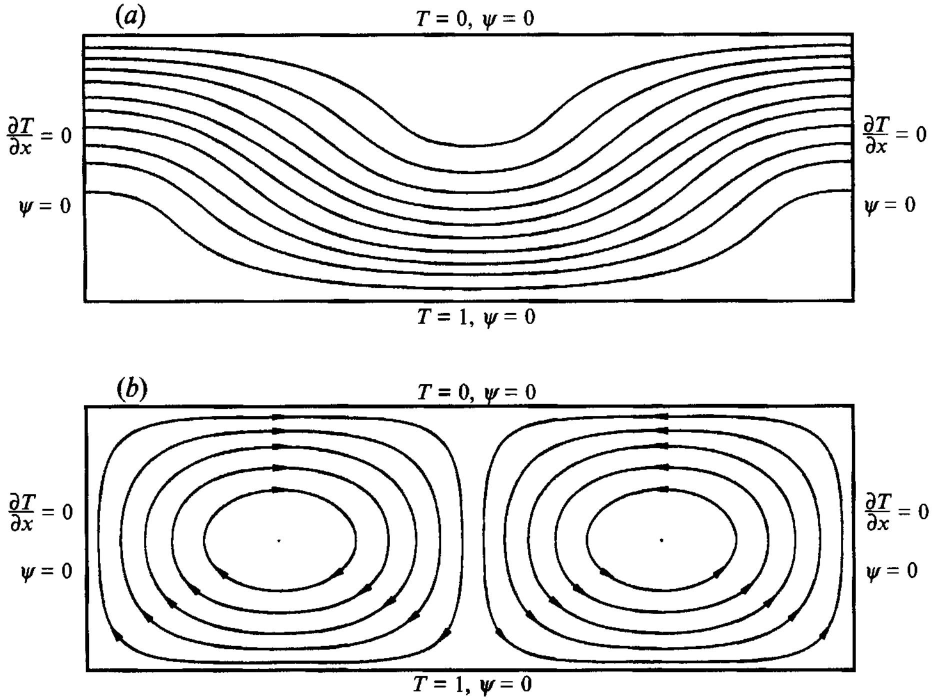

FIGURE 6. Graphical representation of the analytical solutions for the flow and temperature fields corresponding to $\theta_{mn} = 1.57$, $Ra = 50$, $L = 3$, $m = 2$ and $n = 1$. (*a*) 10 isotherms equally divided between $T_{min} = 0$ and $T_{max} = 1$. (*b*) 10 streamlines equally divided between $\psi_{min} = -2.3$ and $\psi_{max} = 2.3$.

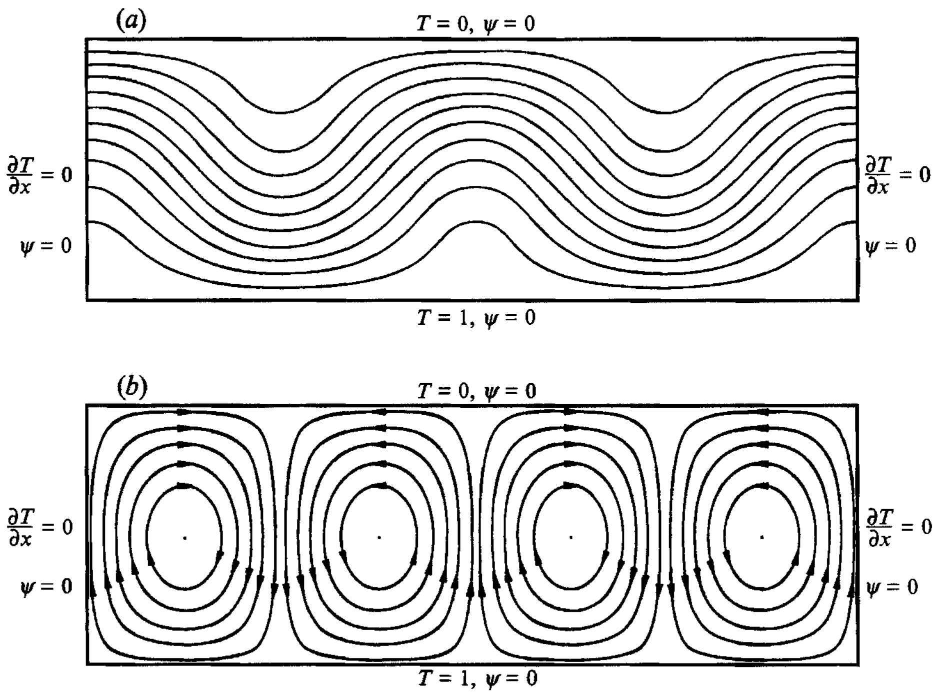

FIGURE 7. As figure 6 but for $\theta_{mn} = 4.25$ and $m = 4$. (*a*) 10 isotherms equally divided between $T_{min} = 0$ and $T_{max} = 1$. (*b*) 10 streamlines equally divided between $\psi_{min} = -1.67$ and $\psi_{max} = 1.67$.

θ_{mn} in the expression for $\lambda(x, z)$ in table 2. A three-convection-cell solution is also possible and can be realized by the initial perturbations that include this mode ($m = 3$). In this case the amplitude is obtained as the solution to the homogeneous amplitude equation, in which $\eta = 0$.

7. Conclusions

Natural convection in horizontal layers or rectangular domains has a marked dependence on the heterogeneity of the porous medium. It was found that the heterogeneous thermal conductivity may cause an unconditional occurrence of natural convection if it does not satisfy a certain form of separation of variables. As no restriction was found on the permeability function, it is concluded that the heterogeneity with respect to the permeability does not affect the motionless condition. The division of the horizontal layer or rectangular domain into vertical columns or horizontal sub-layers represents configurations that were treated in the literature (Gheorghitza 1961; McKibbin & O'Sullivan 1980; Rubin 1981; Gjerde & Tyvand 1984; McKibbin 1986; Nield 1987). These particular cases of horizontal or vertical stratification are examples which satisfy identically the motionless condition. In the work presented here the analytical solutions for rectangular weak heterogeneous porous domains which are heated from below show the direction of the flow to be controlled by the thermal conductivity function. In the particular cases of horizontal or vertical stratification, i.e. the division of the horizontal domain into vertical columns or horizontal sub-layers, the amplitude equation is not affected by the weak heterogeneity of the porous medium.

These conditions have been extensively treated in the literature. For the more general but nevertheless weak stratification it was concluded that within a certain range of slightly supercritical Rayleigh number values a symmetric thermal conductivity function reinforces a symmetrical flow while an antisymmetric function favours an antisymmetric flow. The examples presented here suggest the possible existence of a mechanism for flow pattern selection associated with natural convection occurring in heterogeneous porous domains. Except for the higher-order solutions, the weak heterogeneity with respect to permeability plays a relatively passive role. It does not affect the solutions at the leading order, in contrast to the significant effect resulting from the weak heterogeneity with respect to the effective thermal conductivity.

Appendix

The double Fourier expansion for $\lambda^{(3)}(x, z)$ performed in §4 is expressed in the following form:

$$\lambda^{(3)}(x,z) = \tfrac{1}{4}\lambda_{00}^{cc} + \frac{1}{2}\sum_{i=1}^{\infty}\left[\lambda_{i0}^{cc}\cos\left(\frac{i\pi x}{L}\right) + \lambda_{i0}^{sc}\sin\left(\frac{i\pi x}{L}\right)\right]$$

$$+\frac{1}{2}\sum_{j=1}^{\infty}\left[\lambda_{0j}^{cc}\cos(j\pi z) + \lambda_{0j}^{cs}\sin(j\pi z)\right]$$

$$+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\lambda_{ij}^{cc}\cos\left(\frac{i\pi x}{L}\right)\cos(j\pi z) + \lambda_{ij}^{cs}\cos\left(\frac{i\pi x}{L}\right)\sin(j\pi z)\right.$$

$$\left. + \lambda_{ij}^{sc}\sin\left(\frac{i\pi x}{L}\right)\cos(j\pi z) + \lambda_{ij}^{ss}\sin\left(\frac{i\pi x}{L}\right)\sin(j\pi z)\right], \qquad \text{(A 1)}$$

where the coefficients λ_{ij}^{cc}, λ_{ij}^{cs}, λ_{ij}^{sc} and λ_{ij}^{ss} are defined as follows:

$$\left.\begin{aligned}
\lambda_{ij}^{cc} &= \frac{1}{L}\int_0^1\int_0^L \lambda^{(3)}(x,z)\cos\left(\frac{i\pi x}{L}\right)\cos(j\pi z)\,\mathrm{d}x\,\mathrm{d}z,\\
\lambda_{ij}^{cs} &= \frac{1}{L}\int_0^1\int_0^L \lambda^{(3)}(x,z)\cos\left(\frac{i\pi x}{L}\right)\sin(j\pi z)\,\mathrm{d}x\,\mathrm{d}z,\\
\lambda_{ij}^{sc} &= \frac{1}{L}\int_0^1\int_0^L \lambda^{(3)}(x,z)\sin\left(\frac{i\pi x}{L}\right)\cos(j\pi z)\,\mathrm{d}x\,\mathrm{d}z,\\
\lambda_{ij}^{ss} &= \frac{1}{L}\int_0^1\int_0^L \lambda^{(3)}(x,z)\sin\left(\frac{i\pi x}{L}\right)\sin(j\pi z)\,\mathrm{d}x\,\mathrm{d}z.
\end{aligned}\right\}\tag{A 2}$$

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